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Some Constraction with Q- Function For Coupled Coincidence Point Theorem in Partially Ordered Quasi Metric Spaces

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Abstract : The purpose of this article is to prove coupled coincidence and coupled fixed point theorem for non linear contractive mappings in partially ordered complete quasi - metric spaces using the concept of g monotone mapping with a Q- function q. The presented theorems are generalizations of the recent coupled fixed point theorems due to N. Hussain et al. [17], Bhaskar and Lakshmikantham [12], Lakshmikantham and Ciric [20] and many others. We also give an example in support of our theorem for which result of N. Hussain et al. [17].

Keywords : Coupled fixed point, Coupled Common Fixed Point, Coupled Coincidence point, Mixed monotone mapping, Mixed g-monotone mapping.

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1 Introduction and Preliminaries

The Banach contractive mapping principle [11] is an important result of analysis and it has been applied widely in a number of branches of mathematics. It has been noted that the Banach contraction prin-

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ciple [11] was defined on complete metric space. In the few years ago many researchers found that this contraction mappings is also true in another spaces like Banach spaces, cone metric spaces, Fuzzy metric spaces, G- metric spaces and so on. One of the most interesting spaces that is partially ordered metric spaces was introduced by Matthews in 1994 as a part of denotional semantics of dataflow networks (see [22, 23]). Partial ordered metric spaces play an important role in constructing models in the theory of computation (see [26, 27, 28, 29, 30]) and applied to the periodic boundary value problem for different equations, (see [24, 25, 15, 1, 14, 16, 10, 13, 18]) and the references cited therein. Recently, Bhaskar and Lakshmikantham [12] presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [12] noted that their theorem can be used to investigate a large class of problems and discussed the existence and uniqueness of solution for a periodic boundary value problem. Beside this, Al-Homidan et al [3] introduced the concept of a Q-function defined on a quasi-metric space which generalizes the notions of a τ -function and a ω-distance and establishes the existence of the solution of equilibrium problem (see also [[4, 5, 6, 7, 8]). The aim of this paper is to extend the results of Lakshmikantham and Ciric [20] for a mixed monotone nonlinear contractive mapping in the setting of partially ordered quasi-metric spaces with a Q-function q.We prove some coupled coincidence and coupled common fixed point theorems for a pair of mappings. Our results extend the recent coupled fixed point theorems due to Lakshmikantham and Ciric [20] and many others.

Recall that if (X,$\preceq$) is a partially ordered set and $F : X \to X$ such that for each $x, y \in X, x \preceq y$ implies $F(x) \preceq F(y)$, then a mapping F is said to be non decreasing. Similarly, a non increasing mapping is defined. Bhaskar and Lakshmikantham [12] introduced the following notions of a mixed monotone mapping and a coupled fixed point.

Definition 1.1. *Let (X,$\preceq$) is a partially ordered set and $F : X \to X$. The mapping F is said to have the mixed monotone property if F is nondecreasing monotone in first argument and is a nonincreasing monotone in its second argument, that is, for any $x, y \in X$*

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad &\Longrightarrow F(x_1, y) \preceq F(x_2, y) \\ y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad &\Longrightarrow F(x, y_1) \succeq F(x, y_2) \end{aligned}$$

Definition 1.2. *Let (X,$\preceq$) is a partially ordered set and $F : X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g - monotone property if F is g - nondecreasing monotone in first argument and is a g - nonincreasing monotone in its second argument, that is, for any $x, y \in X$*

$$\begin{aligned} x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad &\Longrightarrow F(x_1, y) \preceq F(x_2, y) \\ y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \quad &\Longrightarrow F(x, y_1) \succeq F(x, y_2) \end{aligned}$$

It is clear that Definition 1.2 reduced to 1.1 when g is the identity mapping.

Definition 1.3. *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if*

$$F(x, y) = x \quad F(y, x) = y$$

Definition 1.4. *An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = g(x) \quad F(y, x) = g(y)$$

It is easy to see that coupled coincidence point can be reduced to coupled fixed point on taking g be an identity mapping.
The main theoretical result of Lakshmikantham and Ciric [20] is the following coupled fixed point theorems

Theorem 1.5. *Let $(X,\preceq)$ be a partially ordered set, and suppose, there is a metric d on X such that (X,d) is a complete metric space. Assume there is a function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(t) < t$ and $lim_{r \to t^+}\psi(r) < t$ for each $t > 0$, and also suppose that $F : X \times X$ and $g : X \to X$ such that F has the mixed g- monotone property and*

$$d(F(x, y), F(u, v)) \quad \preceq \quad \psi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \tag{1.1}$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, and g is continuous and commutes with F, and also suppose that either

(a) F is continuous or

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0), \quad g(y_0) \succeq F(y_0, x_0) \tag{1.2}$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y), \quad g(y) = F(y, x) \tag{1.3}$$

that is F and g have a coupled coincidence.

Definition 1.6. *Let X be a nonempty set. A real valued function $d : X \times X \to R^+$ is said to be quasi metric space on X if*

(M_1) $d(x, y) \succeq 0$ for all $x, y \in X$,

(M_2) $d(x, y) = 0$ if and only if $x = y$,

(M_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$

The pair (X,d) is called a quasi- metric space.

Definition 1.7. *Let (X,d) be a quasi metric space. A mapping $q : X \times X \to R^+$ is called a Q- function on X if the following conditions are satisfied:*

(Q_1) *for all* $x, y, z \in X$,

(Q_2) *if* $x \in X$ *and* $(y_n)_{n\succeq 1}$ *is a sequence in* X *such that it converges to point y (with respect to quasi metric) and* $q(x, y_n) \preceq M$ *for some* $M = M(x)$, *then* $q(x, y) \preceq M$;

(Q_3) *for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $q(z, x) \preceq \delta$ *and* $q(z, y) \preceq \delta$ *implies that* $d(x, y) \preceq \epsilon$.

Remark 1.8. *If (X,d) is a metric space, and in addition to* $(Q_1) - (Q_3)$, *the following condition are also satisfied:*

(Q_4) *for any sequence* $(x_n)_{n\succeq 1}$ *in* X *with* $\lim_{n\to\infty} \sup\{q(x_n, x_m) : m > n\} = 0$ *and if there exist a sequence* $(y_n)_{n\succeq 1}$ *in* X *such that* $\lim_{n\to\infty} q(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} d(x_n, y_n) = 0$

then a Q- function is called τ*-function, introduced by Lin and Du [21]. It has been shown in [21] that every* ω*- function, introduced and studied by Kada et al. [19], is a* τ*- function. In fact, if we consider (X,d) as a metric space and replace* (Q_2) *by the following condition:*

(Q_5) *for any* $x \in X$, *the function* $p(x, .) \to R^+$ *is lower semi continuous,*

then a Q- function is called a ω*- function on X. Several examples of* ω*- functions are given in [19]. It is easy to see that if* $(q(x, .)$ *is lower semi continuous, then (*Q_2*) holds. Hence, it is obvious that every* ω*-function is* τ*- function and every* τ*- function is Q- function, but the converse assertions do not hold.*

Example 1.9. *Let* $X = R$. *Define* $d : X \times X \to R^+$ *by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \mid y \mid & \text{otherwise} \end{cases}$$

and $q : X \times X \to R^+$ *by*

$$q(x, y) = \mid y \mid, \quad \forall x, y \in X.$$

Then one can easily see that d is a quasi- metric space and q is a Q- function on X, but q is neither a τ*-function nor a* ω*- function.*

Example 1.10. *Define* $d : X \times X \to R^+$ *by*

$$d(x, y) = \begin{cases} y - x, & \text{if } x = y \\ 2(x - y) & \text{otherwise} \end{cases}$$

and $q : X \times X \to R^+$ *by*

$$q(x, y) = \mid x - y \mid, \quad \forall x, y \in X.$$

Then one can easily see that d is a quasi- metric space and q is a $Q-$ *function on* X, *but q is neither a* τ*- function nor a* ω*- function, because* (X, d) *is not a metric space.*

The following lemma lists some properties of a $Q-$ function on X which are similar to that of a ω-function (see [19]).

Lemma 1.11. *Let $q : X \times X \to R^+$ be a Q - function on X. Let $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ be sequences in X, and let $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ be such that they converges to 0 and $x, y, z \in X$. Then, the following hold:*

1. *if $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$ for all $n \in N$, then $y = z$. In particular, if $q(x, y) = 0$ and $q(x, z) = 0$ then $y = z$;*
2. *if $q(x_n, y_n) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$ for all $x \in N$, then $\{y_n\}_{n\in N}$ converges to z;*
3. *if $q(x_n, x_m) \preceq \alpha_n$ for all $n, m \in N$ with $m > n$, then $\{x_n\}_{n\in N}$ is a Cauchy sequence ;*
4. *if $q(y, x_n) \preceq \alpha_n$ for all $n \in N$, then $\{x_n\}_{n\in N}$ is a Cauchy sequence ;*
5. *if $q_1, q_2, q_3....q_n$ are Q- functions on X, then $q(x, y) = \max\{q_1(x, y), q_2(x, y), q_3(x, y), \ldots, q_n(x, y)\}$ is also a $Q-$ function on X.*

2 Main Result

Analogous with Definition 1.1, Lakshmikantham and Circ [20] introduced the following concept of the mixed $g-$ monotone mapping.

Definition 2.1. *Let $(X,\preceq)$ is a partially ordered set and $F : X \to X$ and $g : X \to X$. The mapping F is said to has the mixed $g-$monotone property if F is nondecreasing $g-$ monotone in first argument and is a nonincreasing $g-$ monotone in its second argument, that is, for any $x, y \in X$*

$$\begin{aligned} x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad &\Longrightarrow F(x_1, y) \preceq F(x_2, y) \\ y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \quad &\Longrightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned}$$

Note that if g is the identity mapping, then the Definition 2.1 reduces to Definition 1.1.

Definition 2.2. *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

Definition 2.3. *Let X be a non empty set and $F : X \times X \to X$ and $g : X \to X$. one says F and g are commutative if*

$$g(F(x, y) = F(g(x), g(y))$$

for all $x, y \in X$.

The main result of N.Hussain et al. [17] is as follows;

Theorem 2.4. *Let $(X,\preceq, d)$ be a partially ordered complete quasi metric space with a Q-function q on X. Assume that the function $\psi : [0, +\infty) \to [0, +\infty)$ is such that $\psi(t) < t$ for each $t > 0$. Further suppose that $k \in (0, 1)$ and $F : X \times X \to X$, $g : X \to X$ are such that F has the mixed g-monotone property and*

$$q(F(x, y), F(u, v)) \quad \preceq \quad k\psi\left(\frac{q(g(x), g(u)) + q(g(y), g(v))}{2}\right) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, and g is continuous and commutes with F, and also suppose that either

(a) F is continuous or

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0), \quad g(y_0) \succeq F(y_0, x_0) \tag{2.2}$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y), \quad g(y) = F(y, x) \tag{2.3}$$

that is F and g have a coupled coincidence.

Following definition plays an important role to prove of our main result.

Definition 2.5. *Let $(X,\preceq,d)$ be a partially ordered complete quasi metric space with a Q-function q on X. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say F is a generalized g-Meir-Keeler type contraction if, for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$,*

$$\epsilon \preceq \frac{1}{2}[q(g(x), g(u)) + q(g(y), g(v))] < \epsilon + \delta(\epsilon) \Longrightarrow q(F(x, y), F(u, v)) < \epsilon. \tag{2.4}$$

Definition 2.6. *Let Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following*

(i) ψ is continuous and non decreasing,

(ii) $\psi(t) = 0$ if and only if $t = 0$,

(iii) $\psi(t) < t$ for each $t > 0$,

(iv) $\psi(t_1 + t_2) \preceq \psi(t_1) + \psi(t_2)$, $\forall t_1, t_2 \in [0, \infty)$.

Now we give the main result of this paper, which is as follows.

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Suppose that $F : X \times X \to X$; $g : X \to X$ are such that F has the mixed g-monotone property. Assume that $\psi \in \Psi$ such that*

$$q(F(x, y), F(u, v)) \quad \preceq \quad \psi\left(rq(g(x), g(u)) + sq(g(y), g(v))\right) \tag{2.5}$$

for all $x, y, u, v \in X$ and $r, s \in (0, 1)$, $0 < r + s < 1$ for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, and g is continuous and commutes with F, and also suppose that either *(a) F is continuous or*

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0), \quad g(y_0) \succeq F(y_0, x_0) \tag{2.6}$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y), \quad g(y) = F(y, x) \tag{2.7}$$

that is F and g have a coupled coincidence.

Proof. Choose $x_0, y_0 \in X$ to be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0)$ $\succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n), \quad \forall n \tag{2.8}$$

$$\succeq 0 \tag{2.9}$$

We will show that

$$g(x_n) \preceq g(x_{n+1}), \quad \forall n \succeq 0, \tag{2.10}$$

$$g(y_n) \tag{2.11}$$

$$\succeq g(y_{n+1}), \quad \forall n \succeq 0. \tag{2.12}$$

We will use the mathematical induction. Let $n = 0$. Since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0)$ $\succeq F(y_0, x_0)$ and as $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(x_0, y_0)$. Thus 2.10 and 2.11 hold for $n = 0$. Suppose now that 2.10 and 2.11 hold for some fixed $n \succeq 0$. Then, since F has a mixed g- monotone property, from 2.8 and 2.10

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n), \quad (y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}) \tag{2.13}$$

and from 2.8 and 2.11,

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n), \quad F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}) \tag{2.14}$$

Now from 2.13 and 2.14, we get

$$g(x_{n+1}) \preceq g(x_{n+2}), \quad g(y_{n+1}) \succeq g(y_{n+2}) \tag{2.15}$$

Thus, by the mathematical induction, we conclude that 2.8 and 2.10 hold for all $n \succeq 0$.Therefore,

$$\begin{aligned} g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq .... \preceq g(x_n) \preceq g(x_{n+1}) \preceq ..... \\ g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq .... \succeq g(y_n) \succeq g(y_{n+1}) \succeq ..... \end{aligned} \tag{2.16}$$

Denote

$$\delta_n = q(g(x_n), g(x_{n+1})) + q(g(y_n), g(y_{n+1})) \tag{2.17}$$

We show that

$$\delta_n \preceq (r+s)\psi\left(\delta_{n-1}\right) \tag{2.18}$$

Since $g(x_{n-1}) \preceq g(x_n)$ and $g(y_{n-1}) \succeq g(y_n)$, from 2.5 and 2.15 we have

$$q(g(x_n), g(x_{n+1})) = q(F(x_{n-1}, y_{n-1}), F(x_n, y_n))$$

$$q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \quad \preceq \quad \psi\left(rq(g(x_{n-1}), g(x_n)) + sq(g(y_{n-1}), g(y_n))\right) \tag{2.19}$$

$$q(g(x_n), g(x_{n+1})) \quad \preceq \quad \psi\left(rq(g(x_{n-1}), g(x_n)) + sq(g(y_{n-1}), g(y_n))\right) \tag{2.20}$$

Similarly, we have

$$q(g(y_n), g(y_{n+1})) = q(F(y_{n-1}, x_{n-1}), F(y_n, x_n))$$

$$q(g(y_n), g(y_{n+1})) \quad \preceq \quad \psi\left(rq(g(y_{n-1}), g(y_n)) + sq(g(x_{n-1}), g(x_n))\right) \tag{2.21}$$

Adding 2.20 and 2.21 we obtain 2.18. Since $\psi < t$ for $t > 0$, it follows, from 2.18, that

$$0 \preceq \delta_n \preceq (r+s)\delta_{n-1} \preceq (r+s)^2\delta_{n-2} \preceq ...... \preceq (r+s)^n\delta_0 \tag{2.22}$$

as $n \to \infty$ we get

$$\lim_{n\to\infty} \delta_n \quad = \quad 0 \tag{2.23}$$

Thus, ‘

$$\lim_{n\to\infty} [q(g(x_n), g(x_{n+1})) + q(g(y_n), g(y_{n+1}))] \quad = \quad 0 \tag{2.24}$$

Now we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. For $m > n$, and since $\psi(t), t$ for each $t > 0$, we have

$$\begin{aligned}
\delta_{mn} &= q(g(x_n), g(x_m)) + q(g(y_n), g(y_m)) \\
&\preceq [q(g(x_n), g(x_{n+1})) + q(g(y_n), g(y_{n+1}))] \\
&+ [q(g(x_{n+1}), g(x_{n+2})) + q(g(y_{n+1}), g(y_{n+2}))] + ... + [q(g(x_{m-1}), g(x_m)) + q(g(y_{m-1}), g(y_m))] \\
&= \delta_n + \delta_{n+1} + \delta_{n+2} + ... + \delta_{m-1} \\
&\preceq \delta_n + (r+s)\psi\left(\delta_n\right) + (r+s)\psi\left(\delta_{n+1}\right) + ... + (r+s)\psi\left(\delta_{m-2}\right) \\
&\preceq \delta_n + (r+s)\left(\delta_n + \delta_{n+1} + \delta_{n+3} + ... + \delta_{m-2}\right) \\
&\preceq \delta_n + (r+s)(\delta_n + \delta_{n+1} + \delta_{n+2} + ...) \\
&\preceq \delta_n + (r+s)\left(\delta_n + (r+s)\psi\left(\delta_n\right) + (r+s)\psi\left(\delta_{n+1}\right) + ... + (r+s)\psi\left(\delta_{m-2}\right)\right) \\
&\preceq \delta_n + (r+s)(\delta_n + (r+s)\delta_n + (r+s)\delta_{n+1} + ...) \\
&\preceq \delta_n + (r+s)(\delta_n + (r+s)\delta_n + (r+s)^2\delta_n + (r+s)^3\delta_n ....) \\
&= \delta_n(1 + (r+s) + (r+s)^2 + (r+s)^3 + ...) \\
&= \left(\frac{1}{1-(r+s)}\right)\delta_n
\end{aligned}$$

as $n \to \infty$

$$\left(\frac{1}{1-(r+s)}\right)\delta_n \to 0$$

This means that for $m > n > n_0$

$$q(g(x_n), g(x_m)) \preceq \left(\frac{1}{1-(r+s)}\right)\delta_n, \quad q(g(y_n), g(y_m)) \preceq \left(\frac{1}{1-(r+s)}\right)\delta_n \tag{2.25}$$

Therefore by Lemma 1.11 $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete, thee exists $x, y \in X$

$$\lim_{n\to\infty} g(x_n) = x, \quad lim_{n\to\infty} g(y_n) = y \tag{2.26}$$

and 2.26 combined with the continuity of g yields

$$\lim_{n\to\infty} g(x_n) = g(x), \quad lim_{n\to\infty} g(y_n) = g(y) \tag{2.27}$$

From 2.13 and continuity of F and q

$$\begin{aligned} g(g(x_{n+1})) &= g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \\ g(g(y_{n+1})) &= g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \end{aligned} \tag{2.28}$$

We now show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Case -1. Suppose that assumption (a) holds. Taking the limit as $n \to \infty$ in 2.28, and using the continuity of F, we get

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} F(g(x_n), g(y_n)) = F\left(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)\right) = F(x, y), \\ g(y) &= \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} F(g(y_n), g(x_n)) = F\left(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(x_n)\right) = F(y, x). \end{aligned} \tag{2.29}$$

Thus,

$$g(x) = F(x, y), \quad g(y) = F(y, x). \tag{2.30}$$

Case - 2. Suppose that the assumption (b) holds. Let $h(x) = gg(x)$. Now, since g is continuous, $\{g(x_n)\}$ is non decreasing with $g(x_n) \to x$, $g(x_n) \preceq x$ for all $n \in N$, and $\{g(y_n)\}$ is non increasing with $g(y_n) \to y$, $g(y_n) \succeq y$ for all $n \in N$, so $(h(x_n)_{n\succeq 1}$ is non decreasing, that is

$$h(x_0) \preceq h(x_1) \preceq h(x_2) \preceq \ldots\ldots\ldots\ldots. \preceq h(x_n) \preceq h(x_{n+1}) \preceq \ldots. \tag{2.31}$$

with $h(x_n) = gg(x_n) \to g(x)$, $h(x_n) \preceq g(x)$ for all $n \in N$, and $(h(x_n)_{n\succeq 1}$ is non increasing, that is

$$h(y_0) \succeq h(y_1) \succeq h(y_2) \succeq \ldots\ldots\ldots\ldots. \succeq h(x_n) \succeq h(x_{n+1}) \succeq \ldots. \tag{2.32}$$

with $h(y_n) = gg(y_n) \to g(y)$, $h(y_n) \preceq g(y)$ for all $n \in N$. Let

$$\gamma_n = q(h(x_n), h(x_{n+1})) + q(h(y_n), h(y_{n+1})). \tag{2.33}$$

Then replacing g by h and δ by γ in 2.17, we get $\gamma_n \preceq (r+s)\psi(\gamma_{n-1})$ such that $\lim_{n\to\infty} \gamma_n = 0$. We show that

$$\begin{aligned} \lim_{n\to\infty} q(h(x_n), g(x)) + q(h(y_n), g(y)) = 0, \\ \lim_{n\to\infty} q(h(x_n), F(x, y)) + q(h(y_n), F(y, x)) = 0. \end{aligned} \tag{2.34}$$

In δ_{mn}, replacing g by h and δ by γ, we get

$$q(h(x_n),h(x_{n+1}))+q(h(y_n),h(y_{n+1}))\preceq \frac{1}{1-(r+s)}\gamma_n\to 0,\quad as\quad n\to\infty, \tag{2.35}$$

that is, for $m>n>n_0$,

$$q(h(x_n),h(x_m))\preceq \frac{1}{1-(r+s)}\gamma_n,\quad q(h(y_n),h(y_m))\preceq \tfrac{1}{1-(r+s)}\gamma_n \tag{2.36}$$

or for $m>n=n_0+1$,

$$q(h(x_n),h(x_m))\preceq \left(\frac{1}{1-(r+s)}\right)\gamma_{n_0+1},\quad q(h(y_n),h(y_m))\preceq \left(\tfrac{1}{1-(r+s)}\right)(\gamma_{n_0+1}) \tag{2.37}$$

Then, since $h(x_m)\to g(x)$, $h(y_m)\to g(y)$, and $h(x_{n_0+1}),h(y_{n_0+1})\in X$, by axiom (Q_2) of the Q-function, we get

$$q(h(x_{n_0+1}),g(x))\preceq M_{g(x)},\quad q(h(y_{n_0+1}),g(y))\preceq M_{g(y)} \tag{2.38}$$

Therefore by the triangle inequality and 2.38 we have (for $n>n_0$) *Case - 3.*

$$\begin{aligned} q(h(x_n),g(x))+q(h(y_n),g(y)) &\preceq q(h(x_n),g(x_{n+1}))+q(h(y_n),g(y_{n+1}))\\ &+q(h(x_{n+1}),g(x))+q(h(y_{n+1}),g(y)) \end{aligned} \tag{2.39}$$

$$q(h(x_n),g(x))+q(h(y_n),g(y))\preceq \gamma_n+M_{g(x)}+M_{g(y)}$$

This implies that

$$\begin{aligned} q(h(x_n),g(x))&\preceq \gamma_n+M_{g(x)}+M_{g(y)},\\ q(h(y_n),g(y))&\preceq \gamma_n+M_{g(x)}+M_{g(y)}. \end{aligned}$$

Case - 4. Also, we have

$$\begin{aligned} q(h(x_n),F(x,y)) &+ q(h(y_n),F(y,x))\\ &\preceq q(h(x_n),g(x_{n+1}))+q(h(y_n),g(y_{n+1}))\\ &+ q(h(x_{n+1}),F(x,y))+q(h(y_{n+1}),F(y,x))\\ &= \gamma_n+[q(F(g(x_n),g(y_n)),F(x,y)+q(F(g(y_n),g(x_n)),F(y,x)]\\ &\preceq \gamma_n+\psi\left(rq(gg(x_n),g(x))+sq(gg(y_n),g(y))\right)\\ &+ \psi\left(rq(gg(y_n),g(y))+sq(gg(x_n),g(x))\right) \end{aligned}$$

or

$$\begin{aligned}
q(h(x_n), F(x,y)) &+ q(h(y_n), F(y,x)) \\
&= \gamma_n \psi(rq(h(x_n), g(x)) + sq(h(y_n), g(y))) \\
&+ \psi(rq(h(y_n), g(y)) + sq(h(x_n), g(x))) \\
&= \gamma_n 2\psi(rq(h(x_n), g(x)) + sq(h(y_n), g(y))) \\
&\preceq \gamma_n + (r+s)(q(h(x_n), g(x)) + q(h(y_n), g(y)))) \\
&\preceq \gamma_n + (r+s)(\gamma_n + M_{g(x)} + M_{g(y)}) \\
&= \mu\gamma_n where \mu = (r+s)\left(1 + \frac{1}{1-(r+s)} + \frac{1}{(1-(r+s))}\right)
\end{aligned}$$

That is, for $n > n_0$,

$$q(h(x_n), F(x,y)) \preceq \mu\gamma_n, \quad q(h(y_n), F(y,x)) \preceq \mu\gamma_n \tag{2.40}$$

Hence by the Lemma 1.11

$$g(x) = F(x,y), \quad g(y) = F(y,x) \tag{2.41}$$

Thus, F and g have a coupled coincidence point. □

Following example shows that Theorem 2.7 is generalization of Theorem 2.4.

Example 2.8. *Let $X = [0,\infty)$, with the usual partial ordered $\preceq$. Defined $d : X \times X \to R^+$ by*

$$d(x,y) = \begin{cases} y - x, & \text{if } x = y \\ 2(x-y) & \text{otherwise} \end{cases}$$

and $q : X \times X \to R^+$ by

$$q(x,y) = \mid x - y \mid, \quad \forall x, y \in X \tag{2.42}$$

Then d is a quasi metric and q is a Q- function on X. Thus, $(X, d, \preceq)$ is a partially ordered complete quasi metric space with Q- function q on X. Let $\psi(t) = \frac{t}{2}$, for $t > 0$.Defined $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{2x-3y}{6}, & \text{if } x \preceq y \\ 0 & x ¡ 0 \end{cases}$$

and $g : X \to X$ by $g(x) = \frac{6x}{r+s}$ where $0 < r + s < 1$. Then F has the mixed g-monotone property with

$$g(F(x,y)) = \begin{cases} \frac{2x-3y}{r+s}, & \text{if } x \preceq y \\ 0 & x ¡ 0 \end{cases}$$

It is easy to see that

$$g(F(x,y)) = F(g(x), g(y)) \tag{2.43}$$

and F,g are both continuous in their domains and $F(X \times X) \subseteq g(X)$. *Let* $x, y, u, v \in X$ *be such that* $g(x) \preceq g(u)$ *and* $g(y) \succeq g(v)$. *There are four possibilities for 2.5 to hold. We first compute expression on the left 2.5 for these cases:*

Case- [1] $x \succeq y$ *and* $u \succeq v$

$$\begin{aligned} q(F(x,y), F(u,v)) = &| F(x,y) - F(u,v) | \\ &= \quad | \frac{(2x-3y)}{6} - \frac{(2u-3v)}{6} | \\ &= \quad \frac{1}{6} | 2(x-u) - 3(y-v) | \\ &\preceq \quad \frac{1}{3}\{| x-y | + \frac{1}{2} | y-v |\}. \end{aligned}$$

Case- [2] $x \succeq y$ *and* $u < v$

$$\begin{aligned} q(F(x,y), F(u,v)) = &| F(x,y) - 0 | \\ &= \quad | \frac{(2x-3y)}{6} - \frac{(2u-3v)}{6} | \\ &= \quad \frac{1}{3} | (x-u) - \frac{1}{2}(y-v) | \\ &\preceq \quad \frac{1}{3}\{| x-y | + \frac{1}{2} | y-v |\}. \end{aligned}$$

Case- [3] $x < y$ *and* $u \succeq v$

$$\begin{aligned} q(F(x,y), F(u,v)) = &| 0 - F(u,v) | \\ &= \quad | \frac{(2x-3y)}{6} - \frac{(2u-3v)}{6} | \\ &= \quad \frac{1}{3} | (x-u) - \frac{1}{2}(y-v) | \\ &\preceq \quad \frac{1}{3}\{| x-y | + \frac{1}{2} | y-v |\}. \end{aligned}$$

Case- [4] $x < y$ *and* $u < v$

$$q(F(x,y), F(u,v)) = 0$$

On the other hand (in all the above four cases), we have

$$
\begin{aligned}
&\psi\left(rq(g(x),g(u))+sq(g(y),g(v))\right)\\
&\qquad\preceq\ rq(g(x),g(u))+sq(g(y),g(v))\\
&\qquad\preceq\ \frac{(r+s)}{4}\left[\frac{5}{(r+s)}[\mid x-u\mid+\mid y-v\mid]\right]\\
&\qquad\preceq\ \frac{1}{3}\mid x-u\mid+\frac{1}{2}\mid y-v\mid]
\end{aligned}
$$

Thus, f satisfies the condition 2.5 of Theorem 2.7. Now suppose that $(x_n)_{n\succeq 1};(y_n)_{n\succeq 1}$, be respectively non decreasing and non increasing sequences such that $x_n\to x$ and $y_n\to y$, then by Theorem 2.7 $x_n\preceq x$ and $y_n\succeq y$ for all $n\succeq 1$ Let $x_0=0$ and $y_0=6(r+s)$. Then this point satisfies the relations

$$g(x_0)=0=F(x_0,y_0),\ as\quad x_0<y_0 and\quad g(y_0)=36>r+s=F(y_0,x_0) \tag{2.44}$$

Therefore by Theorem 2.7, there exists $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$.

It is easily to see that Example 2.8 is not true for Theorem 2.4.

Corollary 2.9. *Let $(X,\preceq, d)$ be a partially ordered complete quasi metric space with a Q-function q on X. Assume that the function $\psi:[0,+\infty)\to[0,+\infty)$ is such that $\psi(t)<t$ for each $t>0$. Further suppose that $k\in(0,1)$ and $F:X\times X\to X$, $g:X\to X$ are such that F has the mixed g-monotone property and*

$$q(F(x,y),F(u,v))\quad\preceq\quad k\psi\left(\frac{q(g(x),g(u))+q(g(y),g(v))}{2}\right) \tag{2.45}$$

for all $x,y,u,v\in X$ for which $g(x)\preceq g(u)$ and $g(y)\preceq g(v)$. Suppose that $F(X\times X)\subseteq g(X)$, and g is continuous and commutes with F, and also suppose that either

(a) F is continuous or

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\}\to x$ then $x_n\preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\}\to y$ then $y_n\succeq y$ for all n.

If there exists $x_0,y_0\in X$ such that

$$g(x_0)\preceq F(x_0,y_0),\quad g(y_0)\succeq F(y_0,x_0) \tag{2.46}$$

then there exist $x,y\in X$ such that

$$g(x)=F(x,y),\quad g(y)=F(y,x) \tag{2.47}$$

that is F and g have a coupled coincidence.

Proof. It is easily to see that if we take $r=s=\frac{k}{2}$ where $k\in(0,1)$ and from the property of ψ in Theorem 2.7 then we get Corollary 2.9. □

Corollary 2.10. *Let $(X, \preceq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Further, suppose that $r, s \in (0,1)$, $0 < r+s < 1$ and $F : X \times X \to X$; $g : X \to X$ are such that F has the mixed g-monotone property and*

$$q(F(x,y), F(u,v)) \quad \preceq \quad rq(g(x), g(u)) + sq(g(y), g(v)) \tag{2.48}$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, and g is continuous and commutes with F, and also suppose that either

(a) F is continuous or

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0), \quad g(y_0) \succeq F(y_0, x_0) \tag{2.49}$$

then there exist $x, y \in X$ such that

$$g(x) = F(x,y), \quad g(y) = F(y,x) \tag{2.50}$$

that is F and g have a coupled coincidence.

Proof. Taking $\psi(t) = t$ in Theorem 2.7, we obtain 2.10. □

Now, we will prove the existence and uniqueness theorem of a coupled common fixed point, Note that $(S, \preceq)$ is a partial ordered set, then endow the product $S \times S$ with the following partial order;

$$\forall (x,y),(u,v) \in S \times S, \quad (x,y) \preceq (u,v) \Longleftrightarrow x \preceq u, y \succeq v. \tag{2.51}$$

From Theorem 2.7, it follows that the set C(F,g) of coupled coincidence is non empty.

Theorem 2.11. *The hypothesis of Theorem 2.7 holds. Suppose that for every $(x,y),(y^*,x^*) \in X \times X$ there exists a $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(x^*,y^*), F(y^*,x^*))$. Then F and g have a unique common coupled fixed point; that is, there exist a unique $(x,y) \in X \times X$ such that*

$$x = g(x) = F(x,y), \quad y = g(y) = F(y,x) \tag{2.52}$$

Proof. By Theorem 2.7 $C(F,g) \neq \phi$. Let $(x,y),(x^*,y^*) \in C(F,g)$. We show that

$$g(x) = F(x,y), \quad g(y) = F(y,x)$$

and

$$g(x^*) = F(x^*,y^*), \quad g(y^*) = F(y^*,x^*)$$

then

$$g(x) = g(x^*), \quad g(y) = g(y^*) \tag{2.53}$$

By assumption therre is $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ so that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then as the proof of the Theorem 2.7, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n, v_n), \quad g(v_{n+1}) = F(v_n, u_n) \tag{2.54}$$

Further, set $x_0 = x, y_0 = y, x_0^* = x^*, y_0^* = y^*$ and as above, define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x_n^*)\}$ and $\{g(y_n^*)\}$. Then it is easy to show that

$$\begin{aligned} g(x_n) = F(x, y), &\quad g(y_n) = F(y, x), \\ g(x_n^*) = F(x^*, y^*), &\quad g(y_n^*) = F(y^*, x^*) \end{aligned}$$

for all n
$\succeq 1$. Since $(F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y))$ and $(F(u, v), F(v, u)) = (g(u_1), g(v_1))$ are comparable therefore $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$. It is easy to show that $(g(x), g(y))$ and $(g(u_n), g(v_n))$ are comparable, that is, $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$ for all $n \succeq 1$. From 2.5 and property of ψ, we have

$$\begin{aligned} q(g(u_{n+1}), g(x)) &+ q(g(v_{n+1}), g(y)) \\ &= q(F(u_n, v_n), F(x, y)) + q(F(v_n, u_n), F(y, x)) \\ &\preceq \psi(rq(g(u_n), g(x)) + sq(g(v_n), g(y)) \\ &+ \psi(rq(g(v_n), g(y)) + sq(g(u_n), g(x)) \\ &= 2\psi(rq(g(u_n), g(x)) + sq(g(v_n), g(y)) \\ &\preceq (r+s)(rq(g(u_n), g(x)) + sq(g(v_n), g(y)) \\ &\preceq \psi(rq(g(u_{n-1}), g(x)) + sq(g(v_{n-1}), g(y)) \\ &+ \psi(rq(g(v_{n-1}), g(y)) + sq(g(u_{n-1}), g(x)) \\ &= 2\psi(rq(g(u_{n-1}), g(x)) + sq(g(v_{n-1}), g(y)) \\ &\preceq (r+s)^2(rq(g(u_{n-1}), g(x)) + sq(g(v_{n-1}), g(y)) \\ &\preceq \psi(rq(g(u_{n-2}), g(x)) + sq(g(v_{n-2}), g(y)) \\ &+ \psi(rq(g(v_{n-2}), g(y)) + sq(g(u_{n-2}), g(x)) \\ &= 2\psi(rq(g(u_{n-2}), g(x)) + sq(g(v_{n-2}), g(y)) \\ &\preceq (r+s)^3(rq(g(u_{n-2}), g(x)) + sq(g(v_{n-2}), g(y)) \\ &\preceq \ldots\ldots \preceq (r+s)^n(q(g(u_0), g(x)) + q(g(v_0), g(y)) \\ &\quad (r+s)^n t_0 \to 0 \end{aligned}$$

as $n \to \infty$ where $t_0 = q(g(u_0), g(x)) + q(g(v_0), g(y))$. From this it follows that, for each $n \in N$

$$q(g(u_{n+1}, g(x)) \preceq (r+s)^n t_0, \quad q(g(v_{n+1}, g(y)) \preceq (r+s)^n t_0 \tag{2.55}$$

similarly, one can prove that

$$q(g(u_{n+1}, g(x^*)) \preceq (r+s)^n t_0', \quad q(g(v_{n+1}, g(y^*)) \preceq (r+s)^n t_0' \tag{2.56}$$

where $t_0' = q(g(u_0), g(x^*)) + q(g(v_0), g(y^*))$. Thus by Lemma 1.11, $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by commutativity of F and g we have

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)), \quad g(g(y)) = g(F(y, x)) = F(g(y), g(x)), \tag{2.57}$$

Denote $g(x) = z$, $g(y) = w$. Then from 2.57

$$g(z) = F(z, w), \quad g(w) = F(w, z) \tag{2.58}$$

Thus, (z, w) is a coupled coincidence point. Then, from 2.53, with $x^* = z$ and $y^* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$; that is

$$g(z) = z, \quad g(w) = w, \tag{2.59}$$

From 2.58 and 2.59

$$z = g(z) = F(z, w), \quad w = g(w) = F(w, z) \tag{2.60}$$

Therefore, (z, w) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then, by 2.53, we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

Corollary 2.12. *Let $(X, \preceq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Assume that the function $\psi : [0, \infty) \to [0, \infty)$ is such that*

$$\psi(t) < t, \quad \textit{for each} t > 0. \tag{2.61}$$

Further, suppose that $r, s \in (0, 1)$, $\ 0 < r + s < 1$ and $F : X \times X \to X$; such that F has the mixed monotone property and

$$q(F(x, y), F(u, v)) \quad \preceq \quad \psi\left(rq(x, u) + sq(y, v)\right) \tag{2.62}$$

for all $x, y, u, v \in X$ for which $x \preceq u$ and $y \succeq v$.

(a) F is continuous or

(b) X has the following property:

(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,

(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), \quad y_0 \succeq F(y_0, x_0) \tag{2.63}$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \quad y = F(y, x). \tag{2.64}$$

Further, if x_0, y_0 are comparable then $x = y$, that is $x = F(x, y)$

Proof. It is enough if we take $g = I$ (the identity mapping in X) in Theorem 2.7. □

Corollary 2.13. *Let $(X, \preceq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Suppose that $r, s \in (0, 1)$, $0 < r + s < 1$ and $F : X \times X \to X$; such that F has the mixed monotone property and*

$$q(F(x, y), F(u, v)) \quad \preceq \quad rq(x, u) + sq(y, v) \tag{2.65}$$

for all $x, y, u, v \in X$ for which $x \preceq u$ and $y \succeq v$.

(a) F is continuous or

(b) X has the following property:

- *(i) if a non decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all n,*
- *(ii) if a non increasing sequence $\{y_n\} \to y$ then $y_n \succeq y$ for all n.*

If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), \quad y_0 \succeq F(y_0, x_0) \tag{2.66}$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \quad y = F(y, x). \tag{2.67}$$

Further, if x_0, y_0 are comparable then $x = y$, that is $x = F(x, y)$

Proof. Taking $\psi(t) = t$ in Corollary 2.12, we obtain Corollary 2.13. □

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