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Representation of Dirichlet Average of K-Series via Fractional Integrals And Special Functions

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Abstract : The aim of this paper is to investigate the Dirichlet averages of the K- series. Representations for such constructions in two and multi- dimensional cases are derived in term of the Riemann-Liouville fractional integrals and of the hypergeometric functions of several variables. Special cases when the above Dirichlet averages coincide with different type of the Mittag-Leffler functions and hypergeometric functions of one and several variables are obtained.

Keywords : K-series, Mittag-Leffler functions, Dirichlet averages, Riemann-Liouville fractional integrals, Hypergeometric function of one and several variables.

1 Introduction

The K-series defined by Gehlot and Ram [3], as

$$ {}_pK_q^{(\alpha,\delta)_m}(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;z) = {}_pK_q^{(\alpha,\delta)_m}(z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{s=1}^{q}(b_s)_k} \frac{z^k}{\prod_{i=1}^{m}\Gamma(\delta_i k+\alpha_i)} \tag{1.1} $$

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where $a_j, b_s, \alpha_i \in \mathbb{C};\ \delta_i \in \mathbb{R}, (j = 1, ..., p; s = 1, ..., q; i = i, ..., m)$.
The series (1.1) is valid for none of the parameters b_s $(s = 1, ..., q)$ is negative integer or zero. If any parameter a_j $(j = 1, ..., p)$ in (1.1) is zero or negative, the series terminates in to polynomial in z, and

(i) if $p < q + \sum_{i=1}^{m} \delta_i$,then the power series on the right side of (1.1) is absolutely convergent for all $z \in C$,

(ii) if $p = q + \sum_{i=1}^{m} \delta_i$ and $|z| = 1$, then the series is absolutely convergent for all $|z| < \prod_{i=1}^{m} (|\delta_i|)^{\delta_i}$ and $|z| = \prod_{i=1}^{m} (|\delta_i|)^{\delta_i}$, $\mathrm{Re}(\sum_{s=1}^{q} (b_s) + \sum_{i=1}^{m} (\beta_i) - \sum_{j=1}^{p} (a_j)) > \frac{2+q+m-p}{2}$.

When $p = q = 1, a_1 = \rho$ and $b_1 = 1$, (1.1) coincide with the generalized Mittag-Leffler function studied by Kilbas et al. [5],

$$ {}_1K_1^{(\alpha,\delta)_m}(\rho; 1, (\alpha, \delta)_m; z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\prod_{i=1}^{m} \Gamma(\delta_i k + \alpha_i)\Gamma(k + 1)} = E_\rho((\alpha, \delta)_m; z). \tag{1.2} $$

In (1.1), if we set $p = q = 1, a_1 = \rho, b_1 = 1$ and $m = 1$, we get the generalized Mittag-Leffler function studied by Prabhakar [6],

$$ {}_1K_1^{(\alpha,\delta)_m}(\delta; 1, (\alpha, \delta)_1; z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\prod_{i=1}^{m} \Gamma(\delta k + \alpha)\Gamma(k + 1)} = E^{\rho}_{\alpha,\delta}(z). \tag{1.3} $$

For $m = 1, \delta_1 = \alpha$ and $\alpha_1 = 1$ in (1.1), function ${}_pK_q^{(\beta,\eta)_m}(z)$ reduces to the M-Series introduced by Sharma [8]

$$ {}_pK_q^{(\alpha,1)_1}(a_1, ..., a_p; b_1, ..., b_q, (\alpha, 1)_1; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k z^k}{\prod_{s=1}^{q} (b_s)_k \Gamma(\alpha k + 1)} =_p M_q^{\alpha}(z). \tag{1.4} $$

The Dirichlet averages of a function is a certain kind integral averages with respect to Dirichlet measure. The concept of Dirichlet averages was studied by Carlson [2]. A detailed and comprehensive account of various types of Dirichlet averages has been given by Carlson in his monograph [1].
We will need some notations in the further exposition. First, the symbol E_{n-1} will denote the Euclidean simplex in $R^{n-1}, n \geq 2$ defined by

$$ E_{n-1} = \{(u_1, ..., u_{n-1}) : u_1 \geq 0, ..., u_{n-1} \geq 0, u_1 + ... + u_{n-1} \leq 1\}. \tag{1.5} $$

Next, the concept of the Dirichlet average. Let Ω be a convex set in $\mathbb{C}$ and let $z = (z_1, ..., z_n) \in \Omega^n, n \geq 2$ and let f be a measurable function on Ω, then the general Dirichlet averages of a function was defined by Carlson [1] in the form

$$ F(b; z) = \int_{E_{n-1}} f(uoz) d\mu_b(u), \tag{1.6} $$

where $d\mu_b(u)$ is the Dirichlet measure

$$ d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} ... u_{n-1}^{b_{n-1}-1} (1 - u_1 - ... - u_{n-1})^{b_n-1}, \tag{1.7} $$

with the multivariate Beta function

$$B(b)=\frac{\Gamma(b_1)...\Gamma(b_n)}{\Gamma(b_1+...+b_n)},\qquad Re(b_j)>0\ (j=1,...,n) \tag{1.8}$$

and

$$(uoz)=\sum_{j=1}^{n-1}u_jz_j+(1-u_1-...-u_{n-1})z_n. \tag{1.9}$$

For $n=1, F(b;z)=f(z)$. In particular if $n=2$, we have

$$d\mu_{\beta\beta'}(u)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}u^{\beta-1}du. \tag{1.10}$$

Carlson [1] investigated the average (1.6) for $f(z)=z^k$, $k\in R$ in the form

$$R_k(b:z)=\int\limits_{E_{n-1}}(uoz)^k d\mu_b, \tag{1.11}$$

and for $n=2$, Carlson [1, 2] proved that

$$R_k(\beta,\beta';x,y)=\frac{1}{B(\beta,\beta')}\int\limits_0^1[ux+(1-u)y]^k u^{\beta-1}(1-u)^{\beta-1}du, \tag{1.12}$$

where $\beta,\beta'\in\mathbb{C}$, $\min\{Re(\beta),Re(\beta')\}>0$; $x,y\in R$ and $B(\beta,\beta')$ is standard Beta function. The Dirichlet average of the K- series (1.1) is given by

$$M_q^p[(\alpha_i,\delta_i)_{1,m};(\beta,\beta';x,y)]=\int\limits_{E_1}[{}_pK_q^{(\alpha,\delta)_m}(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(uoz)]d\mu_{\beta,\beta'}(u), \tag{1.13}$$

where $\alpha_i,a_j,b_s\in\mathbb{C};\delta_i\in R,(i=1,...,m;j=1,...,p;s=1,...,q)$, $z=(x,y)\in R$, $\{Re(\beta),Re(\beta')\}>0$. We prove the representation for (1.13) in terms of the Riemann-Liouville fractional integral of order $\alpha\in\mathbb{C}$,$Re(\alpha)>0$([7])

$$(I_{a+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int\limits_a^x(x-t)^{\alpha-1}f(t)dt,\qquad (x>a,a\in R), \tag{1.14}$$

and in terms of the Srivastava-Daoust generalization of the Lauricella hypergeometric function in n variables defined by [9]

$$\mathbb{S}_{C:D';...,D^{(n)}}^{A:B';...,B^{(n)}}\begin{pmatrix}[(a):\theta',...,\theta^{(n)}]:[(b'):\varphi'];...;[(b^{(n)}):\varphi^{(n)}] & x_1\\ [(c):\psi',...,\psi^{(n)}]:[(d'):\delta'];...;[(d^{(n)}):\delta^{(n)}] & x_n\end{pmatrix}$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j'+...+m_n\theta_j^{(n)}}\prod\limits_{j=1}^{B'}(b_j')_{m_1\varphi_j'}...\prod\limits_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_1\varphi_j^{(n)}}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi_j'+...+m_n\psi_j^{(n)}}\prod\limits_{j=1}^{D'}(d_j')_{m_1\delta_j'}...\prod\limits_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_1\delta_j^{(n)}}}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}, \tag{1.15}$$

Srivastava and Daoust [10] reported that the series in (1.15) convergence absolutely (i). for all $x_1, ..., x_n \in \mathbb{C}$, when $\Delta_l = 1 + \sum_{j=1}^{C} \psi_j^{(l)} + \sum_{j=1}^{D^{(l)}} \delta_j^{(l)} - \sum_{j=1}^{A} \theta_j^{(l)} - \sum_{j=1}^{B^{(l)}} \varphi_j^{(l)} > 0, l = \overline{1, n}$

(ii). for $|x_1| < \eta_l$ when $\Delta_l = 0, l = \overline{1, n}$, where $\eta_l = \left\{ \mu_l^{1+\sum_{j=1}^{D^{(l)}} \delta_j^{(l)} - \sum_{j=1}^{B^{(l)}} \varphi_j^{(l)}} \frac{\prod_{j=1}^{C} (\sum_{l=1}^{n} \mu_1 \psi_j^{(l)})^{\psi_j^{(l)}} \prod_{j=1}^{D^{(l)}} (\delta_j^{(l)})^{\delta_j^{(l)}}}{\prod_{j=1}^{A} (\sum_{l=1}^{n} \mu_1 \theta_j^{(l)})^{\theta_j^{(l)}} \prod_{j=1}^{B^{(l)}} (\varphi_j^{(l)})^{\varphi_j^{(l)}}} \right\}$

When all $\Delta_l < 0$, $\mathbb{S}^{A:B';...,B^{(n)}}_{C:D';...,D^{(n)}}(x_1, ..., x_n)$ diverges expect at the origin.

The paper is organized as follows. In section 2 we give representation of (1.12) and (1.13) in terms of the Riemann-Liouville fractional integral (1.14). Section 3 is devoted to special cases involving the m- series, generalized Mittag-Leffler functions and the Gauss hypergeometric function. In section 4 we obtain the modification of Dirichlet averages of (1.13) and express special cases. Section 5 deals with representation of Dirichlet averages in terms of Srivastava-Daoust function (1.15) and Dirichlet averages of multivariate function is consider by authors in section 6.

2 Two–Variate Dirichlet Averages

In this section we give representation of (1.12) and (1.13) in terms of the Riemann- Liouville fractional integral (1.14).

Theorem 2.1. *Let $\alpha_i, a_j, b_s, \beta, \beta' \in \mathbb{C}; \delta_i \in R, (i = 1, ..., m; j = 1, ..., p; s = 1, ..., q)$, $min\{Re(\beta), Re(\beta')\} > 0, z = (x, y) \in R$ such that $x > y > 0$ and convergence conditions of the K- series are satisfied. Then the Dirichlet averages of the K- series is given by the formula* $M_q^p[(\alpha_i, \delta_i)_{1,m}; (\beta, \beta'; x, y)]$

$$= \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)(x - y)^{\beta+\beta'-1}} \times \{I_{0+}^{\beta'}(t_p^{\beta-1} K_q^{(\alpha,\delta)_m})(a_1, ..., a_p; b_1, ..., b_q, (\alpha, \delta)_m; (y + t))\}(x - y) \tag{2.1}$$

Proof. According to (1.1) and (1.13), we have

$$H_1 \equiv M_q^p[(\alpha_i, \delta_i)_{1,m}; (\beta, \beta'; x, y)] = \int_{E_1} [{}_pK_q^{(\alpha,\delta)_m}(a_1, ..., a_p; b_1, ..., b_q, (\alpha, \delta)_m; (uoz))] d\mu_{\beta,\beta'}(u)$$

$$= \frac{1}{B(\beta, \beta')} \int_0^1 u^{\beta-1}(1 - u)^{\beta'-1} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k [y + u(x - y)]^k}{\prod_{s=1}^{q} (b_s)_k \prod_{i=1}^{m} (\delta_i k + \alpha_i)} du.$$

By interchanging order of integral and summation, we have

$$H_1 = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{s=1}^{q} (b_s)_k \prod_{i=1}^{m} (\delta_i k + \alpha_i)} \int_0^1 u^{\beta-1}(1 - u)^{\beta'-1}[y + u(x - y)]^k du$$

,

let $u(x-y)=t$,

$$H_1=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\int_0^{x-y}t^{\beta-1}(x-y-t)^{\beta-1}[y+t]^k dt$$

$$=\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^{x-y}\left\{\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k[y+t]^k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\right\}t^{\beta-1}(x-y-t)^{\beta'-1}dt$$

$$=\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\left[\frac{1}{\Gamma(\beta')}\int_0^{x-y}\left\{\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k[y+t]^k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\right\}t^{\beta-1}(x-y-t)^{\beta'-1}dt\right]$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\{I_{0+}^{\beta'}(t_p^{\beta-1}K_q^{(\alpha,\delta)_m})(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(y+t))\}(x-y).$$

This yields (2.1) and thus the Theorem 2.1 is proved. □

3 Special Cases

In this section we consider some particular cases of Theorem 2.1.

Corollary 3.1. *Let the conditions of Theorem 2.1 are satisfied with $m=1,\delta_1=\mu$ and $\alpha_1=1$, then the following result holds* $M_q^p[(1,\mu)_1;(\beta,\beta';x,y)]$

$$\begin{aligned}&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{0+}^{\beta'}\left(t^{\beta-1}{}_pK_q^{(1,\mu)_1}(a_1,...,a_p;b_1,...,b_q,(1,\mu)_1;(y+t)\right)\right\}(x-y)\\&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{0+}^{\beta'}\left(t^{\beta-1}{}_pK_q^{\mu}(a_1,...,a_p;b_1,...,b_q;(y+t)\right)\right\}(x-y).\end{aligned}\tag{3.1}$$

This is the new result for the M-series [8].

Corollary 3.2. *Let the conditions of Theorem 2.1 are satisfied with $p=q=1,a_1=\rho$ and $b_1=1$, then the following result holds* $M_1^1[(\alpha,\delta)_m;(\beta,\beta';x,y)]$

$$\begin{aligned}&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{0+}^{\beta'}\left(t^{\beta-1}{}_1K_1^{(\alpha,\delta)_m}(\rho;1,(\alpha,\delta)_m;(y+t)\right)\right\}(x-y)\\&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{0+}^{\beta'}\left(t^{\beta-1}E_\rho[(\alpha_i,\delta_i)_{1,m};(y+t)\right)\right\}(x-y).\end{aligned}\tag{3.2}$$

This is a new result for the generalized Mittag-Leffler function studied by Kilbas et. al [5].

Corollary 3.3. *Let the conditions of Theorem 2.1 are satisfied with $p=q=1,a_1=\rho,b_1=1$ and $m=1$ then we reach at following result*

$M_1^1[(\alpha,\delta)_1;(\beta,\beta';x,y)]=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{0+}^{\beta'}\left(t^{\beta-1}{}_1K_1^{(\alpha,\delta)_1}(\rho;1,(\alpha,\delta)_1;(y+t)\right)\right\}(x-y)$

Hence,

$$M_{\alpha,\delta}^{\rho}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{0+}^{\beta'}\left(t^{\beta-1}E_{\alpha,\delta}^{\rho}(y+t)\right)\right\}(x-y)\tag{3.3}$$

If we set $\alpha = 1$ in above result (3.3), we come to the

$$M^{\rho}_{1,\delta} = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}} \left\{ I^{\beta'}_{0+} \left(t^{\beta-1} {}_1F_1(\rho;\delta;(y+t)) \right) \right\} (x-y). \quad (3.4)$$

These are the well known results earlier given by Kilbas et al. ([4],p.476-477).

Corollary 3.4. *Let the conditions of Theorem 2.1 are satisfied with $y = 0$, then Theorem 2.1 gives*

$$M^p_q[(\alpha,\delta)_m;(\beta,\beta';x,0)] = \frac{(\beta)_k}{(\beta+\beta')_k} \left({}_pK^{(\alpha,\delta)_m}_q(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;x) \right) \quad (3.5)$$

Corollary 3.5. *Let the conditions of Theorem 2.1 are satisfied with $x = 0$, then Theorem 2.1 gives*

$$M^p_q[(\alpha,\delta)_m;(\beta,\beta';0,y)] = \frac{(\beta)_k}{(\beta+\beta')_k} \left({}_pK^{(\alpha,\delta)_m}_q(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;y) \right) \quad (3.6)$$

4 Modification of the Dirichlet Averages

In this section we consider a modification of the Dirichlet averages $M^p_q[(\alpha_i,\delta_i)_(1,m);(\beta,\beta';x,y)]$ described in (1.13), in the form

$${}_\gamma M^p_q[(\alpha_i,\delta_i)_{1,m};(\beta,\beta';x,y)] = \int\limits_{E_1} (uoz)^{\gamma-1} \left[{}_pK^{(\alpha,\delta)_m}_q(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(uoz)^{\alpha}) \right] d\mu_{\beta,\beta'}(u), \quad (4.1)$$

where $z = (x,y), \gamma \in \mathbb{C}, Re(\gamma) > 0$.

Theorem 4.1. *Let $\alpha_i, a_j, b_s, \beta, \beta' \in \mathbb{C}; \delta_i \in R, (i = 1,...,m; j = 1,...,p; s = 1,...,q)$, $min\{Re(\beta), Re(\beta')\} > 0$, $z = (x,y) \in R$, such that $x > y > 0$ and convergence conditions of the K- series are satisfied. Then for all such that $Re(\gamma) > 0$, ${}_\gamma M^p_q[(\alpha_i,\delta_i)_{1,m};(\beta,\beta';x,y)]$*

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}} \times \left\{ I^{\beta'}_{\gamma+} \left(t^{\gamma-1}(t-y)^{\beta-1} {}_pK^{(\alpha,\delta)_m}_q(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(t)^{\alpha} \right) \right\} (x). \quad (4.2)$$

Proof. By using the equation (1.10) and (4.1), we obtain

$$H_1 \equiv_\gamma M^p_q[(\alpha_i,\delta_i)_{1,m};(\beta,\beta';x,y)] = \int\limits_{E_1} (uoz)^{\gamma-1} [{}_pK^{(\alpha,\delta)_m}_q(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(uoz)^{\alpha})] d\mu_{\beta,\beta'}(u)$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int\limits_0^1 [y+u(x-y)]^{\gamma-1} u^{\beta-1}(1-u)^{\beta'-1} \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_k [y+u(x-y)]^{\alpha k}}{\prod\limits_{s=1}^{q}(b_s)_k \prod\limits_{i=1}^{m}(\delta_i k+\alpha_i)} du.$$

By interchanging order of integral and summation, we have

$$H_1 = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{s=1}^{q}(b_s)_k \prod\limits_{i=1}^{m}(\delta_i k+\alpha_i)} \int\limits_0^1 u^{\beta-1}(1-u)^{\beta'-1}[y+u(x-y)]^{\alpha k+\gamma-1} du$$

,
let $[y+u(x-y)]=t$,

$$H_2=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\int_y^x t^{\alpha k+\gamma-1}(t-y)^{\beta-1}(x-t)^{\beta-1}dt$$

$$=\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_y^x t^{\gamma-1}\left\{\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(t)^{\alpha k}}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\right\}(t-y)^{\beta-1}(x-t)^{\beta'-1}dt$$

$$=\left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\left[\frac{1}{\Gamma(\beta')}\int_y^x\left\{\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(t)^{\alpha k}}{\prod_{s=1}^{q}(b_s)_k\prod_{i=1}^{m}(\delta_i k+\alpha_i)}\right\}(t-y)^{\beta-1}(x-t)^{\beta'-1}dt\right]$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\{I_{y+}^{\beta'}(t^{\gamma-1}(t-y)^{\beta-1}{}_pK_q^{(\alpha,\delta)_m}(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(t)^{\alpha})\}(x).$$

This complete the proof of the Theorem. □

Corollary 4.2. *Let the conditions of Theorem 4.1 are satisfied with $m=1,\delta_1=\mu$ and $\alpha_1=1$, then the following result holds* ${}_{\gamma}M_q^p[(1,\mu)_1;(\beta,\beta';x,y)]$

$$\begin{aligned}&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}{}_pK_q^{(1,\mu)_1}(a_1,...,a_p;b_1,...,b_q,(1,\mu)_1;(t^{\alpha})\right)\right\}(x)\\&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}{}_pK_q^{\mu}(a_1,...,a_p;b_1,...,b_q;(t^{\alpha})\right)\right\}(x).\end{aligned}\tag{4.3}$$

This is the new result for the M-series [8].

Corollary 4.3. *Let the conditions of Theorem 4.1 are satisfied with $p=q=1,a_1=\rho$ and $b_1=1$, then the following result holds* ${}_{\gamma}M_1^1[(\alpha,\delta)_m;(\beta,\beta';x,y)]$

$$\begin{aligned}&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}{}_1K_1^{(\alpha,\delta)_m}(\rho;1,(\alpha,\delta)_m;(t^{\alpha})\right)\right\}(x)\\&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\times\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}E_{\rho}[(\alpha_i,\delta_i)_{1,m};(t^{\alpha})\right)\right\}(x).\end{aligned}\tag{4.4}$$

This is a new result for the generalized Mittag-Leffler function studied by Kilbas et. al [5].

Corollary 4.4. *Let the conditions of Theorem 4.1 are satisfied with $p=q=1,a_1=\rho,b_1=1$ and $m=1$ then we reach at following result*
${}_{\gamma}M_1^1[(\alpha,\delta)_1;(\beta,\beta';x,y)]=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}{}_1K_1^{(\alpha,\delta)_1}(\rho;1,(\alpha,\delta)_1;(t^{\alpha})\right)\right\}(x)$
Hence,

$${}_{\gamma}M_{\alpha,\delta}^{\rho;\alpha}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}E_{\alpha,\delta}^{\rho,\alpha}(t^{\alpha})\right)\right\}(x)\tag{4.5}$$

If we set $\alpha=1$ in above result (4.5), we come to the

$${}_{\gamma}M_{1,\delta}^{\rho,1}=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{y+}^{\beta'}\left(t^{\gamma-1}(t-y)^{\beta-1}{}_1F_1(\rho;\delta;(t)\right)\right\}(x).\tag{4.6}$$

These are the well known results earlier given by Kilbas et al. ([4],p.480).

5 Representation of Dirichlet Averages in terms of Srivastava-Daoust Function

In this section we consider an another kind characterization of the modified Dirichlet averages of the K-series with $\gamma = \rho$ in (4.1) presented as the following result:

Theorem 5.1. *Let* $\alpha_i, a_j, b_s, \beta, \beta' \in \mathbb{C}; \delta_i \in R, (i = 1, ..., m; j = 1, ..., p; s = 1, ..., q)$, $min\{Re(\beta), Re(\beta')\} > 0$, $z = (x, y) \in R$, *such that* $x > y > 0$ *and convergence conditions of the K- series are satisfied. Then for all such that* $Re(\gamma) > 0$, ${}_\rho M_q^p[(1,1)(\alpha_i, \delta_i)_{2,m}; (\beta, \beta'; x, y)]$

$$= \frac{y^{\delta-1}}{\prod_{i=2}^{m}\Gamma(\alpha_i)} \mathbb{S}^{1:p;1}_{0:q:m;1}\left(\begin{matrix}[1-\delta:-\alpha,1]:[(a_j):1]_{1,p};[\beta:1];y^{\alpha},\left(1-\frac{x}{y}\right)\\ -:[1-\delta:-\alpha];[(b_n):1]_{1,q};[(\alpha):\delta]_{2,m};[\beta+\beta':1]\end{matrix}\right) \tag{5.1}$$

Proof. Using (4.1) and the integral representation (1.12), we have

$$H_3 \equiv_\rho M_q^p[(1,1)(\alpha_i, \delta_i)_{2,m}; (\beta, \beta'; x, y)]$$
$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1 [y+u(x-y)]^{\rho-1}u^{\beta-1}(1-u)^{\beta'-1}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k[y+u(x-y)]^{\alpha k}}{\prod_{s=1}^{q}(b_s)_k\prod_{i=2}^{m}(\delta_i k+\alpha_i)k!}du.$$

By interchanging order of integral and summation, we have

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(y)^{\alpha k+\rho-1}}{\prod_{s=1}^{q}(b_s)_k\prod_{i=2}^{m}(\delta_i k+\alpha_i)k!}\int_0^1\left[1-\left(1-\frac{x}{y}\right)u\right]^{\alpha k+\delta-1}u^{\beta-1}(1-u)^{\beta'-1}du$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(y)^{\alpha k+\rho-1}}{\prod_{s=1}^{q}(b_s)_k\prod_{i=2}^{m}(\delta_i k+\alpha_i)k!}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}{}_2F_1\left(\beta,1-\alpha k-\rho;\beta+\beta';\left(1-\frac{x}{y}\right)\right)$$

$$= y^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(y^{\alpha})^k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=2}^{m}(\delta_i k+\alpha_i)k!}{}_2F_1\left(\beta,1-\alpha k-\rho;\beta+\beta';\left(1-\frac{x}{y}\right)\right)$$

$$= y^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k(y^{\alpha})^k}{\prod_{s=1}^{q}(b_s)_k\prod_{i=2}^{m}(\delta_i k+\alpha_i)k!}\sum_{r=0}^{\infty}\frac{(\beta)_r(1-\alpha k-\rho)_r}{(\beta+\beta')_r r!}\left(1-\frac{x}{y}\right)^r$$

Now Since $(1-\alpha k-\rho)_r=\frac{\Gamma(1-\alpha k-\rho+r)}{\Gamma(1-\alpha k-\rho)}=\frac{(1-\rho)_{-\alpha k+r}}{(1-\rho)_{-\alpha k}}$, and thus

$$H_3=y^{\rho-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{s=1}^{q}(b_s)_k}\frac{(1-\rho)_{-\alpha k+r}(y^{\alpha})^k(\beta)_r\left(1-\frac{x}{y}\right)^r}{(1-\rho)_{-\alpha k}\prod_{i=2}^{m}(\delta_i k+\alpha_i)(\beta+\beta')_r r!k!}$$

$$=\frac{y^{\rho-1}}{\prod_{i=2}^{m}\Gamma(\alpha_i)}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1-\rho)_{-\alpha k+r}\prod_{j=1}^{p}(a_j)_k}{(1-\rho)_{-\alpha k}\prod_{s=1}^{q}(b_s)_k}\frac{(y^{\alpha})^k(\beta)_r\left(1-\frac{x}{y}\right)^r}{\prod_{i=2}^{m}(\alpha_i)_{\delta_i k}(\beta+\beta')_r r!k!}$$

$$=\frac{y^{\delta-1}}{\prod_{i=2}^{m}\Gamma(\alpha_i)}\mathbb{S}^{1:p;1}_{0:q:m;1}\left(\begin{array}{c}[1-\delta:-\alpha,1]:[(a_j):1]_{1,p};[\beta:1];y^{\alpha},\left(1-\frac{x}{y}\right)\\ -:[1-\delta:-\alpha];[(b_n):1]_{1,q};[(\alpha):\delta]_{2,m};[\beta+\beta':1]\end{array}\right)$$

This complete the Theorem 5.1. □

6 Dirichlet Averages of Multivariate Function

Let us make the convention that,(λ) denotes the n-tuple of $\lambda_1,...,\lambda_n$. The Dirichlet averages M_q^p and its modification ${}_{\gamma}M_q^p$ are discussed here, where the complex variable vector is $(z)=(z_1,...,z_n)\in\mathbb{C}$ and the prescribed parameters vector are $(d_1,...,d_n)$. Our results are based on the following preliminary assertion:

Lemma 6.1 ([4],p.483, Lemma 1)**.** *Let n be a positive integer, d_j,r_j be complex numbers such that $Re(d_j)>0, Re(r_j)>-1$. Let E_{n-1} be the Euclidean simplex (1.5) and stands for the Dirichlet measure (1.7), then there holds the formula*

$$\int\limits_{E_{n-1}}u_1^{r_1}...u_{n-1}^{r_{n-1}}(1-u_1-...-u_{n-1})^{r_n}d\mu_d(u)=\frac{(d_1)_{r_1}...(d_n)_{r_n}}{(d_1+...+d_n)_{r_1+...+r_n}}. \tag{6.1}$$

The Lauricella functions F_D defined for complex parameters $d\in\mathbb{C}^n$ in term of the multiple series [11] is defined by

$$F_D(a,(d);c;z)=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a)_{m_1+...+m_n}(d_1)_{m_1}...(d_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{z_1^{m_1}...z_n^{m_n}}{m_1!...m_n!} \tag{6.2}$$

Series (6.2) converges for all variables inside unit circle, that is for $\max_{1\le j\le n}|z_j|<1$. Let us remind the Srivastava-Daoust generalization $\mathbb{S}(u)$ of the Lauricella F_D. Now, we will study the following Dirichlet average

$${}_{\rho}M_q^p[(\alpha_i,\delta_i)_{1,m};((d);(1-z))]=\int\limits_{E_{n-1}}(1-uoz)^{\rho-1}[{}_pK_q^{(\alpha,\delta)_m}(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(1-uoz)^{\alpha})]d\mu_d(u). \tag{6.3}$$

we will also need the directly verified formula

$$(1-z_1-...-z_n)^{\eta}=\sum_{r_1,...,r_n=0}^{\infty}(-\eta)_{r_1+...+r_n}\frac{z_1^{r_1}...z_n^{r_n}}{r_1!...r_n!},\quad(|z_1+...+z_n|<1). \tag{6.4}$$

Theorem 6.2. *Let* $\alpha_i, a_j, b_s, \beta, \beta' \in \mathbb{C}; \delta_i \in R, (i = 1, ..., m; j = 1, ..., p; s = 1, ..., q)$, $min\{Re(\beta), Re(\beta')\} > 0$, $z = (x, y) \in R$, *such that* $x > y > 0$ *and convergence conditions of the K- series are satisfied. Then there holds the following formula* ${}_\rho M_q^p[(1,1)(\alpha_i,\delta_i)_{2,m};((d);(1-z))]$

$$= \frac{1}{\prod\limits_{i=2}^{m}\Gamma(\alpha_i)} \mathbb{S}^{0:p;1:(1)}_{2:q:m;0}\begin{pmatrix} -:[(a_j):1]_{1,p}:[\rho:\alpha];[(d):1];1,(-z_1,...,-z_n) \\ [\rho:\alpha;(-1)]:[\sum\limits_{i=1}^{n}(d_n):0;1];[(b_n):1]_{1,q};[(\alpha):\delta]_{2,m};- \end{pmatrix} \tag{6.5}$$

Proof. Consider equation (6.3), we have

$$\begin{aligned} H_4 &\equiv {}_\rho M_q^p[(\alpha_i,\delta_i)_{1,m};((d);(1-z))] \\ &= \int\limits_{E_{n-1}} (1-uoz)^{\rho-1}[{}_pK_q^{(\alpha,\delta)_m}(a_1,...,a_p;b_1,...,b_q,(\alpha,\delta)_m;(1-uoz)^{\alpha})]d\mu_d(u) \\ &= \int\limits_{E_{n-1}} (1-uoz)^{\rho-1}\sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k(1-uoz)^{\alpha k}}{\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\delta_i k+\alpha_i)k!}d\mu_d(u) \\ &= \sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\delta_i k+\alpha_i)k!}\int\limits_{E_{n-1}}(1-uoz)^{\alpha k+\rho-1}d\mu_d(u) \end{aligned}$$

Applying the Lemma 6.1 (6.1) and the polynomial expansion (6.4), letting $|u_1z_1 + ... + u_nz_n| < 0$, to H_4 we obtain that

$$H_4 = \sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\delta_i k+\alpha_i)k!}\sum_{r_1,...,r_n=0}^{\infty}(1-\alpha k-\rho)_{r_1+...+r_n}\frac{z_1^{r_1}...z_n^{r_n}}{r_1!...r_n!}\int\limits_{E_{n-1}}u_1^{r_1}...u_n^{r_n}(1-u_1-...-u_{n-1})^{r_n}d\mu_d(u)$$

$$= \sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\delta_i k+\alpha_i)k!}\sum_{r_1,...,r_n=0}^{\infty}\frac{(1-\alpha k-\rho)_{r_1+...+r_n}(d_1)_{r_1}...(d_n)_{r_n}}{(d_1+...+d_n)_{r_1+...+r_n}}\frac{z_1^{r_1}...z_n^{r_n}}{r_1!...r_n!}$$

the n- fold inner sum (with respect to $r_1, ..., r_n$) forms a Lauricella F_D function, so

$$H_4 = \sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k}{\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\delta_i k+\alpha_i)k!}F_D[(1-\alpha k-\rho);(d);d_1+...+d_n;(z)]$$

,

since, $(1-\alpha k-\rho)_{r_1+...+r_n} = (-1)^{r_1+...+r_n}\frac{(\rho)_{\alpha k}}{(\rho)_{\alpha k-r_1-...-r_n}}$, by applying above transformation to H_4, we arrive at

$$\begin{aligned} H_4 &= \frac{1}{\prod\limits_{i=2}^{m}}\Gamma(\alpha_i)\sum_{k,(r)=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_k(\rho)_{\alpha k}(d_1)_{r_1}...(d_n)_{r_n}(-z_1)^{r_1}...(-z_n)^{r_n}}{(\rho)_{\alpha k-r_1-...-r_n}\prod\limits_{s=1}^{q}(b_s)_k\prod\limits_{i=2}^{m}(\alpha_i)_{\delta_i k}(d_1+...+d_n)k!r_1!...r_n!} \\ &= \frac{1}{\prod\limits_{i=2}^{m}\Gamma(\alpha_i)}\mathbb{S}^{0:p;1:(1)}_{2:q:m;0}\begin{pmatrix} -:[(a_j):1]_{1,p}:[\rho:\alpha];[(d):1];1,(-z_1,...,-z_n) \\ [\rho:\alpha;(-1)]:[\sum\limits_{i=1}^{n}(d_n):0;1];[(b_n):1]_{1,q};[(\alpha):\delta]_{2,m};- \end{pmatrix} \end{aligned}$$

□

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